Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 320 (2009) 691–704
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On eigenproblem solution of damped vibrations associated with gyroscopic moments

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Received 8 January 2008; received in revised form 20 June 2008; accepted 11 August 2008
Handling Editor: M.P. Cartmell
Available online 24 September 2008

Abstract

A new efficient approach is presented for solving the quadratic eigenvalue problem of weakly, nonproportionally damped vibration systems. In the analysis of these systems, gyroscopic moments and external damping are both considered. Traditional restriction of symmetry of inertia, damping and stiffness matrices is slightly relaxed. A second-order perturbation theory is developed such that the perturbed solution is based on the eigensolution of an unperturbed subproblem. This subproblem considers the unperturbed system in two different forms: (i) a conservative, gyroscopic part of an original problem, or (ii) a nonconservative, gyroscopic part of an original problem that is proportionally damped. To cope with asymmetry of the system matrices, a Duncan's like state formulation is used to bring these matrices into a suitable form for perturbations. Two numerical examples are introduced for explaining the detailed implementation of the presented approach. Additionally, a practical problem of rotor supported by two tilting pad-bearings is investigated. The eigensolutions obtained by the current approach match, to a great extent, other solutions obtained by time-consuming exact methods. The investigation procedure given here gives a framework to handle vibration problems of weakly nonproportional damping and/or weakly asymmetric inertia, damping and stiffness matrices.

1. Introduction

The eigenvalue problem is the heart of the linear vibration theory, and its solution provides the vibration analyst with rich foundation about the behavior of this system from the stability and response points of view. This is why the eigenvalue problem is always being under focus of continuous, intensive research activity everywhere. In absence of the dissipative forces, generally, the linear dynamic systems possess classical normal modes [1]. In other words, they have a complete set of real orthogonal eigenvectors that can transform the system into a diagonal form. This form is very delicate for applying the powerful modal superposition method to response calculations. So many structure problems are lightly damped, and can be assumed to have symmetric damping matrix proportional to symmetric mass and stiffness matrices. The self-adjoint

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.08.018

eigensolution is then an easy task to achieve by powerful tools [2], because the system classical normal modes are conserved for proportionally damped systems. These tools become unacceptable even for lightly damped systems having symmetric viscous damping matrices of distribution dissimilar to that of symmetric mass and stiffness matrices. The system is then called nonclassically damped, and response predictions urge using other techniques. A common procedure in the analysis of such systems is to neglect the off-diagonal elements of the associated modal damping matrix. Some other methods for modal and response calculations of nonclassically damped are available in the literature [3–6].

In modern vibration practices [7,8], active damping and fully active vibration control techniques, normally, lead to asymmetric damping and stiffness matrices. Moreover, introducing circulatory forces and gyroscopic moments can further complicate the eigenvalue problem, as it becomes quadratic and asymmetric one. This necessitates the use of other methods like the pioneering Duncan's formulation [9] in which the concept of trivial identity was introduced by Duncan to linearize the problem. But, before going to algorithms that counts on Duncan's formulation or any other methods as in Refs. [10–13], asymmetric systems might possess classical normal modes, and must be checked for their existence. Thus one can avoid complexity in computations and consumption in time, especially, in large-scale models. Conditions under which classical normal modes exist in asymmetric systems are presented in Refs. [14–16].

First- and second-order perturbation techniques have been proven effective in both eigensolution calculations and eigensolution reanalysis problems [17–20]. Meirovitch and Ryland [21] made a second-order perturbation theory developed for the generalized eigensystem $\lambda\mathbf{u} = \mathbf{A}\mathbf{u}$, fruitful for application to lightly damped gyroscopic systems with symmetric mass, damping and stiffness matrices. Chung and Lee [22] extended the theory for application to the generalized eigenproblem $\mathbf{B}\mathbf{u} = \lambda\mathbf{A}\mathbf{u}$ of heavy, but weakly nonproportional damped systems. Although the matrix $\mathbf{A}$ in the basic perturbation theory and the matrices $\mathbf{A}$ and $\mathbf{B}$ in its extension have no restriction except that they must be real, simplifications are necessary to make this theory attractive for application to large-scale systems where hundreds or thousands of degrees of freedom can be considered. Basically, the theory requires the calculation of eigenvalues and right and left eigenvectors of the unperturbed system. This will be the gate for any simplification to be significant. However, the reader is referred to further readings concerning the eigensolution problems in Refs. [23–26].

This paper contributes to the problem of finding approximate eigensolution of asymmetric systems by using the second-order perturbation theory. Firstly, the perturbed solution is based on the solution of unperturbed conservative system formulated in a highly standard eigenvalue problem of single, symmetric positive definite matrix. The perturbed solution is also considered in another form where it represents the nonconservative, gyroscopic part of an original problem that is proportionally damped. Numerical examples will be presented to demonstrate the method in a detailed manner.

2. New formulation

Consider the free vibration problem of a general linear discrete system described by vector differential equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + (\mathbf{C} + \mathbf{G})\dot{\mathbf{q}}(t) + (\mathbf{K} + \mathbf{H})\mathbf{q}(t) = 0 \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n \times n$ real asymmetric matrices. $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{G}$ is an $n \times n$ real skew symmetric gyroscopic matrix, $\mathbf{H}$ is an $n \times n$ real skew symmetric circulatory matrix, and $\mathbf{q}(t)$ is a real $n \times 1$ vector of generalized coordinates. Note here that $\mathbf{G}$ is of conservative nature, while $\mathbf{H}$ is a dissipative one. If the trivial identity

$$(\mathbf{K} + \mathbf{H})\dot{\mathbf{q}} - (\mathbf{K} + \mathbf{H})\dot{\mathbf{q}} = 0 \tag{2}$$

adjoins Eq. (1), the $2n$ associated eigenvalue problem and its adjoint will be

$$\mathbf{B}\,\mathbf{u}_i = \lambda_i \mathbf{A}\,\mathbf{u}_i, \quad \mathbf{B}^{\mathrm{T}}\mathbf{v}_i = \lambda_i \mathbf{A}^{\mathrm{T}}\mathbf{v}_i, \quad i = 1, 2, \ldots, 2n \tag{3}$$

where $\mathbf{q} = e^{\lambda t}\mathbf{u}$ is substituted into Eqs. (1) and (2) for exponential form solutions, λ_i is the ith eigenvalue, $\mathbf{u}_i$ and $\mathbf{v}_i$ are the corresponding right and left eigenvectors, respectively, of the non-self-adjoint eigenvalue problem (3). The biorthogonality of right and left eigenvectors provides

$$\mathbf{v}_j^{\mathrm{T}}\mathbf{A}\,\mathbf{u}_i = \mathbf{u}_j^{\mathrm{T}}\mathbf{A}\,\mathbf{v}_i = 2a_i\delta_{ij}$$
$$\mathbf{v}_j^{\mathrm{T}}\mathbf{B}\,\mathbf{u}_i = \mathbf{u}_j^{\mathrm{T}}\mathbf{B}\,\mathbf{v}_i = 2a_i\lambda_i\delta_{ij}, \quad i,j = 1,2,...,2n \tag{4}$$

where a_i is the scale factor of the ith eigenvector, δ_{ij} is the Kronecker delta. $\mathbf{A}$ and $\mathbf{B}$ are real asymmetric matrices defined by

$$\mathbf{A} = \begin{bmatrix} \mathbf{K}+\mathbf{H} & 0 \\ 0 & \mathbf{M} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & -\mathbf{K}-\mathbf{H} \\ \mathbf{K}+\mathbf{H} & \mathbf{C}+\mathbf{G} \end{bmatrix} \tag{5}$$

Since any real asymmetric matrix can be regarded as a summation of two real matrices one of them symmetric and the other one is skew symmetric, the asymmetric matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ can be written as follows:

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_g, \quad \mathbf{C} = \mathbf{C}_0 + \mathbf{C}_g, \quad \mathbf{K} = \mathbf{K}_0 + \mathbf{K}_g \tag{6}$$

where $\mathbf{M}_0$, $\mathbf{C}_0$ and $\mathbf{K}_0$ are symmetric matrices, and $\mathbf{M}_g$, $\mathbf{C}_g$ and $\mathbf{K}_g$ are skew symmetric ones. For instance, the calculated symmetric and skew symmetric parts of the damping matrix are:

$$\mathbf{C}_0 = (\mathbf{C}+\mathbf{C}^{\mathrm{T}})/2, \quad \mathbf{C}_g = (\mathbf{C}-\mathbf{C}^{\mathrm{T}})/2 \tag{7}$$

It should be mentioned here that the skew symmetric matrix $\mathbf{C}_g$ represents the conservative part of the damping matrix [27,28]. Normally, the true damping is contained into the symmetric part $\mathbf{C}_0$ of the asymmetric damping matrix $\mathbf{A}_0 = \begin{bmatrix} \mathbf{K}_0 & 0 \\ 0 & \mathbf{M}_0 \end{bmatrix}$ [27,28]. It will be further assumed that $\mathbf{M}_0$ and $\mathbf{K}_0$ are positive definite. For perturbation purposes, if Eq. (6) is substituted into Eq. (5), one can write the matrices in Eq. (5) as

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \tag{8}$$

where $\mathbf{A}_0$ and $\mathbf{B}_0$ are considered as unperturbed matrices, and $\mathbf{A}_1$ and $\mathbf{B}_1$ are considered as perturbation matrices. The matrices $\mathbf{A}$ and $\mathbf{B}$ are then called the perturbed matrices. An order of magnitude condition is considered here [22], which states that the elements of the matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ are one order of magnitude smaller than the elements of $\mathbf{A}_0$ and $\mathbf{B}_0$. The following formulation is suggested for the matrices in Eq. (8):

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{K}_0 & 0 \\ 0 & \mathbf{M}_0 \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} 0 & -\mathbf{K}_0 \\ \mathbf{K}_0 & \mathbf{G}+\mathbf{C}_g \end{bmatrix} \tag{9}$$

$$\mathbf{A}_1 = \begin{bmatrix} -\mathbf{K}_g^{\mathrm{T}}-\mathbf{H}^{\mathrm{T}} & 0 \\ 0 & \mathbf{M}_g \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & \mathbf{K}_g^{\mathrm{T}}+\mathbf{H}^{\mathrm{T}} \\ \mathbf{K}_g+\mathbf{H} & \mathbf{C}_0 \end{bmatrix} \tag{10}$$

where $\mathbf{K}_0$ and $\mathbf{M}_0$ are assumed symmetric positive definite matrices, $\mathbf{A}_0$ and $\mathbf{B}_0$ will be symmetric positive definite and skew symmetric, respectively. While the matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ are skew symmetric and symmetric nonnegative definite, respectively. The reason for suggesting that new matrix formulation of Eqs. (9) and (10) is that the unperturbed and perturbation matrices are either symmetric or skew symmetric. In other words, the skew symmetric matrix $\mathbf{A}_1$ is a perturbation to the symmetric matrix $\mathbf{A}_0$ while the symmetric matrix $\mathbf{B}_1$ is a perturbation to the skew symmetric matrix $\mathbf{B}_0$. This permits taking advantages of this arrangement in the perturbation process as will be shown later on. Note also here that the unperturbed matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ represent the conservative gyroscopic part of the original perturbed system $\mathbf{A}$ and $\mathbf{B}$.

The difference between the matrix formulation presented in this section and the most commonly used formulation in the literature, e.g. Chung and Lee [22], can be easily recognized by comparing Eqs. (5), (9) and (10) with Eqs. (A.5), (A.6) and (A.7) in Appendix A.

3. General perturbation theory

In this section, we summarize the second-order perturbation method as was originally developed by Meirovitch and Ryland [21] and modified by Chung and Lee [22] for the second-order perturbation formula to based on the unnormalized eigenvectors rather than the normalized eigenvectors. It will be shown in the forthcoming section that neither the original perturbation results [21] nor the modified ones [22] are suitable for application to the perturbation problem defined by Eqs. (9) and (10) and that a significant modification of the second-order perturbation theory is needed in order to make it applicable to the problem of Eqs. (9) and (10).

The unperturbed eigenvalue problem is assumed to have known eigensolution. In general, the accuracy of the perturbation process is pertinent to the accuracy of the unperturbed solution. The unperturbed eigenproblem and its adjoint one can be expressed as follows:

$$\mathbf{B}_0\mathbf{u}_{0i} = \lambda_{0i}\mathbf{A}_0\mathbf{u}_{0i}, \quad \mathbf{B}_0^{\mathrm{T}}\mathbf{v}_{0i} = \lambda_{0i}\mathbf{A}_0^{\mathrm{T}}\mathbf{v}_{0i}, \quad i = 1, 2, \ldots, 2n \tag{11}$$

where λ_{0i} is the ith eigenvalue, and $\mathbf{u}_{0i}$ and $\mathbf{v}_{0i}$ are ith right and left eigenvectors, respectively. The biorthogonality property of the right and left eigenvectors satisfies the following relations:

$$\begin{aligned} \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} &= \mathbf{u}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{v}_{0i} = 2a_i\delta_{ij} \\ \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} &= \mathbf{u}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{v}_{0i} = 2a_i\lambda_{0i}\delta_{ij}, \quad i,j = 1, 2, \ldots, 2n \end{aligned} \tag{12}$$

To produce the perturbed eigenvalues in terms of the unperturbed ones, one can express the solution of the perturbed eigenvalues as follows:

$$\lambda_i = \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \cdots, \quad i = 1, 2, \ldots, 2n \tag{13a}$$

$$\mathbf{u}_i = \mathbf{u}_{0i} + \mathbf{u}_{1i} + \mathbf{u}_{2i} + \cdots, \quad i = 1, 2, \ldots, 2n \tag{13b}$$

$$\mathbf{v}_i = \mathbf{v}_{0i} + \mathbf{v}_{1i} + \mathbf{v}_{2i} + \cdots, \quad i = 1, 2, \ldots, 2n \tag{13c}$$

The order of any particular term in Eq. (13) is characterized by the first subscript. For example, λ_{1i}, $\mathbf{u}_{1i}$ and $\mathbf{v}_{1i}$ are one order of magnitude smaller than λ_{0i}, $\mathbf{u}_{0i}$ and $\mathbf{v}_{0i}$, respectively. Substituting Eqs. (8) and (13) into Eq. (4) gives, after collection by order of magnitude, the perturbation systems as summarized in (Eqs. (B.1)–(B.3)) Appendix B. The first-order perturbations $\mathbf{u}_1$ and $\mathbf{v}_1$ can be expressed as a linear combinations of $\mathbf{u}_0$ and $\mathbf{v}_0$, respectively, because they span the same space:

$$\mathbf{u}_{1i} = \sum_{k=1}^{2n} \varepsilon_{ik}\mathbf{u}_{0k}, \quad \mathbf{v}_{1i} = \sum_{k=1}^{2n} \gamma_{ik}\mathbf{v}_{0k}, \quad i = 1, 2, \ldots, 2n \tag{14a,b}$$

where ε_{ik} and γ_{ik} are small first-order coefficients. Similarly, the eigenvectors $\mathbf{u}_2$ and $\mathbf{v}_2$ can be expressed as

$$\mathbf{u}_{2i} = \sum_{k=1}^{2n} \tilde{\varepsilon}_{ik}\mathbf{u}_{0k}, \quad \mathbf{v}_{2i} = \sum_{k=1}^{2n} \tilde{\gamma}_{ik}\mathbf{v}_{0k} \quad i = 1, 2, \ldots, 2n \tag{15a,b}$$

where $\tilde{\varepsilon}_{ik}$ and $\tilde{\gamma}_{ik}$ are small second-order coefficients. The solutions for first- and second-order perturbation problems are summarized in Appendix B. Eqs. (B.4) and (B.5) solve for the first-order perturbation problem while Eqs. (B.6)–(B.8) solve for the second-order one.

4. New perturbation results

The conservative gyroscopic system that is represented by the matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ in Eq. (9) is considered as the unperturbed system of equations. If the outcomes of the eigenvalue problem (11) of the unperturbed system (9) satisfy the orthogonality conditions (12), then one can say that the first- and second-order perturbation solutions (Eqs. (B.4)–(B.8)) are possible. Unfortunately, the results of the second-order perturbation theory derived by Chung and Lee [22], although they are quite general with a single restriction that $\mathbf{A}$ and $\mathbf{B}$ must be real, are not liable for application to the unperturbed system (11). In other words, the solution of the unperturbed (11) with the matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ as given in Eq. (9), violates the orthogonality arrangements as given by Eq. (12) and, consequently, mismatches the formulation requirements of the second-order perturbation theory. The task now is to modify this theory to make it liable for application to unperturbed systems like the one considered in this study. The following theorem will clarify this issue.

Theorem 1. *The solution of the unperturbed eigenproblem* (11) *violates the biorthogonality property of the right and left eigenvectors in Eq.* (12)*. Thus, the solution of the first- and second-order perturbation problems is not possible by using the results of the general perturbation theory in Appendix B unless: (i)* $-\mathbf{u}_{0j}$ *replaces* $\mathbf{v}_{0j}$ *in Eqs.* (B.4) *and* (B.5) *and (ii) the sign is reversed at the right-hand sides of Eqs.* (B.6)–(B.8).

Proof. Since $\mathbf{A}_0$ is symmetric positive definite and $\mathbf{B}_0$ is skew symmetric, the eigenvalues of the unperturbed eigensystem (11) will be pure imaginary complex conjugate pairs and the eigenvectors will also be complex conjugate pairs with the following properties [21]:

$$\lambda_{0_{2r}} = \bar{\lambda}_{0_{2r-1}}, \quad r = 1, 2, \ldots, n \tag{16}$$

$$\mathbf{u}_{0_{2r}} = \bar{\mathbf{u}}_{0_{2r-1}}, \quad r = 1, 2, \ldots, n \tag{17}$$

$$\mathbf{v}_{0_r} = \bar{\mathbf{u}}_{0_r}, \quad r = 1, 2, \ldots, 2n \tag{18}$$

where $\bar{\lambda}_0$ and $\bar{\mathbf{u}}_0$ are the complex conjugate of λ_0 and $\mathbf{u}_0$, respectively. Eq. (18) indicates that the left eigenvectors are exactly the complex conjugates of the right eigenvectors. This is due to the nature of the unperturbed eigensystem in which $\mathbf{A}_0$ is symmetric and $\mathbf{B}_0$ is skew symmetric. Also, Eq. (18) simply states that there is no necessity to solve the unperturbed eigenvalue problem twice to have right and left eigenvectors because they are complex conjugates. Now consider the biorthogonality related Eq. (12) upon which the results of the general second-order perturbation theory in the preceding section are derived. And consider an unperturbed eigenproblem of order $2n = 2$ having two eigenvalues, two right eigenvectors and two left eigenvectors. Taking into consideration that the unperturbed system is conservative, and upon using Eqs. (17) and (18), the following orthogonality conditions hold true:

$$\begin{aligned} &\mathbf{u}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} = 2a_1\delta_{11}, \quad \mathbf{u}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} = 0 \\ &\mathbf{u}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} = 0, \quad \mathbf{u}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} = 2a_2\delta_{22} \end{aligned} \tag{19}$$

$$\begin{aligned} \mathbf{v}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} &= \bar{\mathbf{u}}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} = \mathbf{u}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} = 0 \\ \mathbf{v}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} &= \bar{\mathbf{u}}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} = \mathbf{u}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} = 2a_2\delta_{12} \\ \mathbf{v}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} &= \bar{\mathbf{u}}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} = \mathbf{u}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{01} = 2a_1\delta_{21} \\ \mathbf{v}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} &= \bar{\mathbf{u}}_{02}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} = \mathbf{u}_{01}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{02} = 0 \end{aligned} \tag{20}$$

If one considers the Kronecker product properties

$$\delta_{ij} = \begin{cases} 1 \ \text{for} \ i = j \\ 0 \ \text{for} \ i \neq j \end{cases} \tag{21}$$

for application to Eqs. (19) and (20), it follows that the results of the biorthogonality multiplications in Eq. (19), if arranged in a matrix form, lead to a diagonal matrix, while the multiplications in Eq. (20) will lead to a matrix of zero elements. On the basis of this result, one can conclude that using the left eigenvector $\mathbf{v}_0$ in the biorthogonality relations does not justify the arrangements of Eq. (12), and hence a mismatch occurs in the formulation of the second-order perturbation theory leading to incorrect computations if the solution results (Eq. (B.4) through (B.8)) are used in their current form. *This proves the first part of the* Theorem 1. As a result to this, $-\mathbf{u}_0$ should replace $\mathbf{v}_0$ in the formulation starting with Eq. (13c), which will be modified to

$$\mathbf{v}_i = \bar{\mathbf{u}}_{0i} + \bar{\mathbf{u}}_{1i} + \bar{\mathbf{u}}_{2i} + \cdots \tag{22}$$

Or in a more convenient form to the perturbation theory:

$$\mathbf{v}_i = -\mathbf{u}_{0i} - \tilde{\mathbf{u}}_{1i} - \tilde{\mathbf{u}}_{2i} - \cdots, \quad r = 1, 2, \ldots, 2n \tag{23}$$

$$\tilde{\mathbf{u}}_{1i} = \sum_{k=1}^{2n} \gamma_{ik}\mathbf{u}_{0k}, \quad \tilde{\mathbf{u}}_{2i} = \sum_{k=1}^{2n} \tilde{\gamma}_{ik}\mathbf{u}_{0k}, \quad r = 1,2,\ldots,2n \tag{24a,b}$$

The assumption made to develop Eqs. (22)–(24) is mainly based on the nature of the unperturbed system (11) in which the left eigenvectors are the complex conjugates of the right eigenvectors. If one substitutes Eqs. (8), (13a), (13b) and (23) into Eq. (4) and collects by the order of magnitudes, then three sets of problems of different perturbation orders result:

$$\begin{aligned}\tilde{\mathbf{O}}(0):\ &-\mathbf{u}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} = 2a_i\delta_{ij}\\ &-\mathbf{u}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 2a_i\lambda_{0i}\delta_{ij}, \quad i,j = 1,2,\ldots,2n\end{aligned} \tag{25}$$

$$\begin{aligned}\tilde{\mathbf{O}}(1):\ &-\mathbf{u}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{1i} - \mathbf{u}_{0j}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{0i} - \tilde{\mathbf{u}}_{1j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} = 0\\ &-\mathbf{u}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{1i} - \mathbf{u}_{0j}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} - \tilde{\mathbf{u}}_{1j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 2a_i\lambda_{1i}\delta_{ij}, \quad i,j = 1,2,\ldots,2n\end{aligned} \tag{26}$$

$$\begin{aligned}\tilde{\mathbf{O}}(2):\ &-\mathbf{u}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{2i} - \mathbf{u}_{0j}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{1i} - \tilde{\mathbf{u}}_{1j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{1i} - \tilde{\mathbf{u}}_{1j}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{0i} - \tilde{\mathbf{u}}_{2j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} = 0\\ &-\mathbf{u}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{2i} - \mathbf{u}_{0j}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{1i} - \tilde{\mathbf{u}}_{1j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{1i} - \tilde{\mathbf{u}}_{1j}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} - \tilde{\mathbf{u}}_{2j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 2a_i\lambda_{2i}\delta_{ij}, \quad i,j = 1,2,\ldots,2n\end{aligned} \tag{27}$$

where $\tilde{\mathbf{O}}(0)$, $\tilde{\mathbf{O}}(1)$ and $\tilde{\mathbf{O}}(2)$ indicate the modified zero-, first- and second-order perturbation problems, respectively. Substituting Eqs. (14a), (15a) and (24a) into Eq. (26), with orthogonality relations like (19) being utilized, the first-order perturbation solution is provided as in Eqs. (B.4) and (B.5) except that $-\mathbf{u}_{0j}$ replaces $\mathbf{v}_{0j}$ in these equations. Similarly, the substitution of Eqs. (14b), (15b) and (24b) into Eq. (27), and upon the use of orthogonality relations as in Eq. (19), the second-order solution will be the same as that in Eqs. (B.6)–(B.8), except that the sign of all terms at the right-hand sides of these equation is reversed. This completes the proof of Theorem 1. □

It is worth to mention that Theorem 1 in this study is specifically developed for application to undamped unperturbed systems of gyroscopic nature with symmetric $\mathbf{A}_0$ and $\mathbf{B}_1$ matrices and skew symmetric $\mathbf{A}_1$ and $\mathbf{B}_0$ matrices. Furthermore, $\mathbf{A}_0$ should be positive definite. With any other formulation, the reader might switch back to the general theory as derived in Refs. [21,22].

5. Simplified unperturbed calculations

Although a contribution is made to the second-order perturbation theory in the preceding section, the vibration analyst is still in need to a powerful tool by which an unperturbed eigensolution can be systematically generated and a considerable save in time can be ultimately achieved for large-scale systems. Once again, the special form of the unperturbed conservative gyroscopic system can be utilized. Meirovitch [29,30] has shown that a conservative gyroscopic eigensystem like the one of Eq. (11) can be transformed into a highly standard eigenvalue problem of a single, real, positive definite symmetric matrix. The resulting eigenvalues and eigenvectors of this problem will be real. So many fast, efficient algorithms are available for solving the later problem. The procedure of transformation is as follows.

Consider the following unperturbed eigenvalue problem for a conservative gyroscopic system where the eigenvalues are normally pure imaginary:

$$-\mathbf{B}_0\mathbf{u}_0 = \mathbf{i}\omega_0\mathbf{A}_0\mathbf{u}_0, \quad \mathbf{A}_0^{\mathrm{T}} = \mathbf{A}_0 > 0, \quad \mathbf{B} = -\mathbf{B}^{\mathrm{T}} \tag{28}$$

The complex eigensolution of Eq. (28) can be expressed as

$$\begin{matrix} s_r \\ \bar{s}_r \end{matrix} = \pm\mathbf{i}\omega_{0r}, \quad \begin{matrix} \mathbf{u}_{0_r} \\ \bar{\mathbf{u}}_{0_r} \end{matrix} = \mathbf{x}_{0_r} \pm \mathbf{i}\mathbf{y}_{0_r}, \quad r = 1,2,\ldots,n \tag{29}$$

To transform the problem from complex to a real form $\mathbf{u}_0 = \mathbf{x}_0 + \mathbf{i}\mathbf{y}_0$ is substituted into Eq. (28). Then both the real and imaginary part on both sides are equated to give

$$-\mathbf{B}_0\mathbf{x}_0 = \omega_0\mathbf{A}_0\mathbf{y}_0, \quad +\mathbf{B}_0\mathbf{y}_0 = \omega_0\mathbf{A}_0\mathbf{x}_0 \tag{30a,b}$$

Solving Eqs. (25a,b) together, provides

$$\mathbf{B}_0^* x_0 = \lambda_0 \mathbf{A}_0 \mathbf{x}_0, \quad \mathbf{B}_0^* \mathbf{y}_0 = \lambda_0 \mathbf{A}_0 \mathbf{y}_0, \quad \lambda_0 = \omega_0^2 \tag{31}$$

where

$$\mathbf{B}_0^* = \mathbf{B}_0^{\mathrm{T}} \mathbf{A}_0^{-1} \mathbf{B}_0 \tag{32}$$

is a symmetric positive definite matrix. Since $\mathbf{A}_0$ is symmetric positive it can be decomposed by Cholesky decomposition as follows:

$$\mathbf{A}_0 = \mathbf{Q}^{\mathrm{T}} \mathbf{Q} \tag{33}$$

where $\mathbf{Q}$ is a $2n \times 2n$ nonsingular, orthogonal matrix such that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. By using the linear transformation

$$\mathbf{Q}\,\mathbf{x} = \mathbf{z}_x, \quad \mathbf{Q}\,\mathbf{y} = \mathbf{z}_y \tag{34}$$

the eigenvalue problem of Eq. (31) can be reduced to the following standard real one:

$$\mathbf{A}_0^* \mathbf{z} = \lambda_0 \mathbf{z}, \quad \lambda_0 = \omega_0^2 \tag{35}$$

where the last two equations implicitly means that $\mathbf{z}_x$ is the same as $\mathbf{z}_y$, and

$$\mathbf{A}_0^* = (\mathbf{Q}^{\mathrm{T}})^{-1} \mathbf{B}_0^* \mathbf{Q}^{-1} = (\mathbf{Q}^{-1})^{\mathrm{T}} \mathbf{B}_0^* \mathbf{Q}^{-1} \tag{36}$$

The two eigenvalue problems (31) and (35) have the same eigenvalues with each eigenvalue of $\mathbf{A}_0$* retains the multiplicity of two. This multiplicity is expressed as in Eq. (16). By analogy with Eq. (34), the real and imaginary part of $\mathbf{u}_{0_r}$ can be expressed as follows:

$$\mathbf{x}_{0_r} = \mathbf{Q}^{-1}\,\mathbf{z}_{2r-1}, \quad \mathbf{y}_{0_r} = \mathbf{Q}^{-1}\,\mathbf{z}_{2r}, \quad r = 1, 2, \ldots, n \tag{37}$$

Thus the complex eigensolution of Eq. (24), can be reconstructed from the solution of a highly standard eigenvalue problem of single, symmetric positive definite matrix with real eigenvalues and eigenvectors. This, of course, leads to a marginal reduction in the computational time. This reduction becomes more effective as the order of the problem increases. Another idea for future work is that the first- and second-order perturbation solutions in Eq. (B.4) through (B.8) can be related directly to the calculated real eigenvectors (37) rather than reconstructing the complex eigenvector from these real ones. This will save a great part in core of the computer use.

6. Results and discussions

A little problem is to be highlighted first. A hard condition has to be met by any eigenvalue problem in order to be solved by the perturbation method developed in this paper. The entries of the perturbation matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ should be one order of magnitude smaller than the entries of the unperturbed matrices $\mathbf{A}_0$ and $\mathbf{B}_0$. This implicitly means, according to Eqs. (9) and (10) that the entries of the symmetric damping matrix $\mathbf{C}_0$ should be one order of magnitude smaller than the entries of the skew symmetric damping matrix $\mathbf{C}g$. The situation is hard to meet for some applications. Consequently, the theory will not be applicable for those applications. To avoid such a situation, the symmetric damping matrix $\mathbf{C}_0$ can be divided as follows:

$$\mathbf{C}_0 = \mathbf{C}_{0_p} + \mathbf{C}_{0_{np}} \tag{38}$$

where $\mathbf{C}_{0_p}$ is the part of the symmetric damping matrix $\mathbf{C}_0$ that is proportional to the distribution of the mass and stiffness matrices. While, C_{0np}, is the nonproportional part that will replace $\mathbf{C}_0$ in Eq. (10). By the well-known techniques, the proportional part adds to the stiffness and mass matrices in Eq. (1) as it can be expressed as a linear combination of them. Thus, using Eq. (38) will provide a good opportunity for applications, incapable of meeting the hard condition explained above, to be solved by the perturbation technique developed in this paper.

Example 1. The following hypothetical two-dof system is used to show the accuracy of solutions obtained by the current method in comparison with those obtained by exact methods. This example slightly violates the condition that the entries of $\mathbf{A}_1$ and $\mathbf{B}_1$ are one order of magnitude smaller than the entries of the matrices $\mathbf{A}_0$

and $\mathbf{B}_0$. This is just to show that the method presented here is capable of producing acceptable solutions even when this hard condition is violated. The matrices in Eq. (1) are given by

$$\mathbf{M} = \begin{bmatrix} 5 & 2 \\ 3 & 6.5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0.5 \\ 0.3 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & -4 \\ +4 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

According to Eqs. (6) and (7), the resulting $n \times n$ formulation matrices are:

$$\mathbf{M}_0 = \begin{bmatrix} 5 & 2.5 \\ 2.5 & 6.5 \end{bmatrix}, \quad \mathbf{M}_g = \begin{bmatrix} 0 & -0.5 \\ +0.5 & 0 \end{bmatrix}, \quad \mathbf{C}_0 = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}$$

$$\mathbf{C}_g = \begin{bmatrix} 0 & +0.1 \\ -0.1 & 0 \end{bmatrix}, \quad \mathbf{K}_0 = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{K}_g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The $2n \times 2n$ formulation matrices in Eqs. (8)–(10) are then given by

$$\mathbf{A}_0 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 2.5 \\ 0 & 0 & 2.5 & 6.5 \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \\ +4 & 0 & 0 & -3.9 \\ 0 & +5 & +3.9 & 0 \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & +0.5 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0.4 \\ +1 & 0 & 0.4 & 1 \end{bmatrix}$$

Comparing the entries of the matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ with those of the matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ one notices that they violate the condition as mentioned above. The solution results are shown in Table 1. Note that the exact solution of asymmetric eigenproblem are obtained by the using the prepackaged Matlab code *eig* which implements the generalized Schur decomposition algorithm. The computed damping ratios for the two modes are actually contained into the symmetric damping matrix $\mathbf{C}_0$, and are given by $\zeta_1 = 0.086$ and $\zeta_2 = 0.207$, respectively. The results show a significant matching between the second-order perturbation results and those obtained by exact methods. Accuracy to the third decimal is achieved by the current method when compared to the exact one even with the order of magnitude condition is violated.

In Table 1, the results of the method agrees to the second or the third decimal with the results of the exact one. While in Table 2, the agreement between the two solutions is only to the second decimal. The accuracy of

Table 1
Eigenvalues obtained by perturbation and by exact methods

Exact solution	$\tilde{\mathbf{O}}(0)$	$\tilde{\mathbf{O}}(0)+\tilde{\mathbf{O}}(1)$	$\tilde{\mathbf{O}}(0)+\tilde{\mathbf{O}}(1)+\tilde{\mathbf{O}}(2)$
$-0.1262 \pm 1.4548i$	$\pm 1.4736i$	$-0.1287 \pm 1.4736i$	$-0.1287 \pm 1.4545i$
$-0.1267 \pm 0.5963i$	$\pm 0.5924i$	$-0.1266 \pm 0.5924i$	$-0.1266 \pm 0.5962i$

Table 2
Eigenvalues obtained by perturbation and by exact methods

Exact solution	$\tilde{\mathbf{O}}(0)$	$\tilde{\mathbf{O}}(0)+\tilde{\mathbf{O}}(1)$	$\tilde{\mathbf{O}}(0)+\tilde{\mathbf{O}}(1)+\tilde{\mathbf{O}}(2)$
$-0.7220 \pm 2.2559i$	$\pm 2.2842i$	$-0.7244 \pm 2.2842i$	$-0.7248 \pm 2.2840i$
$-0.2780 \pm 0.6681i$	$\pm 0.6678i$	$-0.2755 \pm 6648i$	$-0.2775 \pm 0.6681i$

solutions is constrained by two factors: (i) how small is the elements of the perturbation matrices relative to the unperturbed matrices in the formulation, and (ii) the amount of damping in the system that is under investigation. Keeping the elements of the matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ one order of magnitude smaller than the elements of the matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ will increase the accuracy of solutions [21,22]. In Example 2, the slight violation was made by the damping matrix $\mathbf{C}_0$ and the situation was totally resolved by dividing this matrix into proportional and nonproportional parts as stated by Eq. (38). While the violation in Example 1 was caused by the matrix $\mathbf{H}$ and in such circumstances there no way but to make sure that the violation is slight in order to get a reasonable solution. There is no one rule that can be generalized for estimating how the violation is slight because the situation differs from one problem to another. However, the main contribution of this paper was accomplished by providing a powerful tool for solving eigenproblems which satisfy the magnitude condition. The method developed here is like any other perturbation method finds most of its impact for lightly or moderately damped systems. Fortunately, most of the practical problem in mechanics and structures are lightly or moderately damped systems. The accuracy of solutions gets worse for heavily damped systems with wide nonproportionality of damping.

Example 2. This example shows how to handle nonproportionally damped systems in cases, where the matrix $\mathbf{C}_0$ does not justify the order of magnitude condition. The matrices in Eq. (1) are given by

$$\mathbf{M} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 3 & -1.4 \\ 1.6 & 4 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & -4 \\ +4 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 & -0.5 \\ +0.5 & 0 \end{bmatrix}$$

According to Eqs. (6) and (7), the resulting $n \times n$ formulation matrices are:

$$\mathbf{M}_0 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{M}_g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C}_0 = \begin{bmatrix} 3 & 0.1 \\ 0.1 & 4 \end{bmatrix}$$

$$\mathbf{C}_g = \begin{bmatrix} 0 & -1.5 \\ 1.5 & 0 \end{bmatrix}, \quad \mathbf{K}_0 = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \quad \mathbf{K}_g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

According to Eq. (38), with proportionality constants $\alpha = \beta = 0.35$, the proportional and nonproportional parts of the matrix $\mathbf{C}_0$ are given by

$$\mathbf{C}_{0_p} = \begin{bmatrix} 2.8 & 0 \\ 0 & 3.85 \end{bmatrix}, \quad \mathbf{C}_{0_{np}} = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.15 \end{bmatrix}$$

The $2n \times 2n$ formulation matrices in Eqs. (8)–(10) are then given by

$$\mathbf{A}_0 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \\ +4 & 0 & 2.8 & -5.5 \\ 0 & +7 & +5.5 & 3.85 \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ +0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & +0.5 \\ 0 & 0 & -0.5 & 0 \\ 0 & -0.5 & 0.2 & 0.1 \\ +0.5 & 0 & 0.1 & 0.15 \end{bmatrix}$$

The calculated damping ratios of the two modes are $\zeta_1 = 0.38$ and $\zeta_2 = 0.30$, respectively. The results are shown in Table 2. The accuracy achieved in this example is also considerable.

Example 3. This example is devoted to a practical problem. A rotor shaft, supported on two identical, tilting five-pad bearings with the bearing load acting between pads, is considered in this example as shown in Fig. 1. The bearing data are such that the preload factor is taken 0.66, the length is 0.025 m, the diameter is 0.05 m,

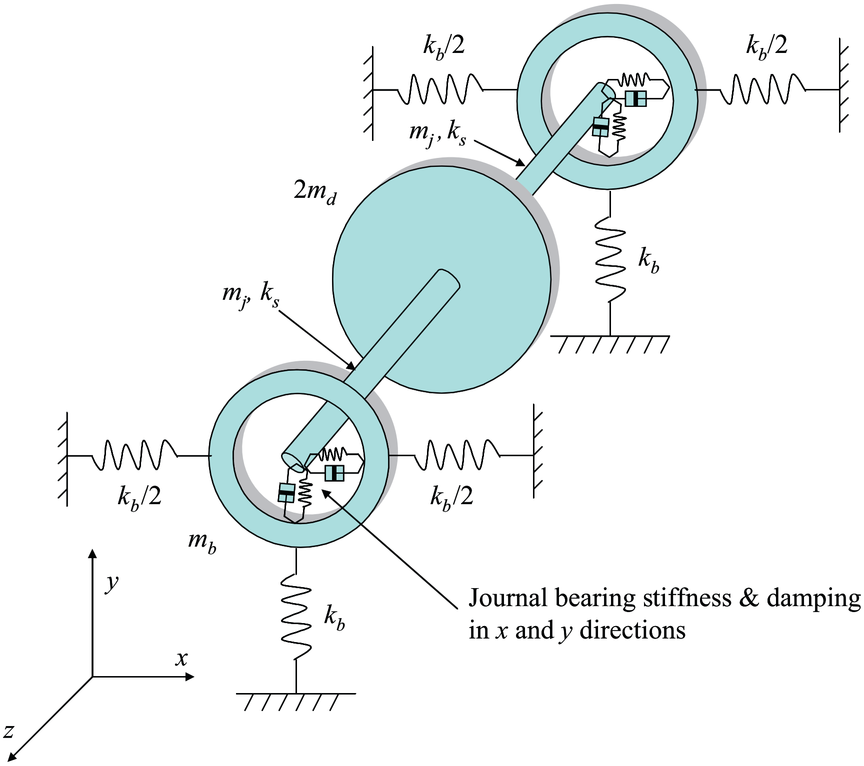

Fig. 1. Rotor-bearing system model.

Table 3
Eigenvalues obtained by perturbation and by exact methods[a]

Exact solution	$\tilde{\mathbf{O}}(0)$	$\tilde{\mathbf{O}}(0)$	$\tilde{\mathbf{O}}(0)+\tilde{\mathbf{O}}(1)+\tilde{\mathbf{O}}(2)$
$-0.0004\pm0.1441i$	$-0.0011\pm0.1440i$	$-0.0004\pm0.1440i$	$-0.0004\pm0.1441i$
$-0.0003\pm0.1496i$	$-0.0012\pm0.1496i$	$-0.0003\pm0.1496i$	$-0.0003\pm0.1496i$
$-0.0211\pm0.6182i$	$-0.0190\pm0.6159i$	$-0.0211\pm0.6159i$	$-0.0213\pm0.6183i$
$-0.0118\pm0.6815i$	$-0.0231\pm0.6789i$	$-0.0121\pm0.6789i$	$-0.0124\pm0.6811i$
$-0.0485\pm0.9460i$	$-0.0453\pm0.9500i$	$-0.0485\pm0.9500i$	$-0.0483\pm0.9462i$
$-0.1175\pm1.1264i$	$-0.0646\pm1.1345i$	$-0.1172\pm1.1345i$	$-0.1169\pm1.1294i$

[a]All numbers in the table should be multiplied by 10^3.

the radial clearance is 0.001 m, and the lubricant viscosity is 0.069 N s/m. The bearing stiffness and damping coefficients are then taken by interpolation from the tabulated coefficients by Someya [31]. The disk mass (per bearing) 150.03 kg, the journal mass is 141.47 kg, the bearing-support mass is 100.8 kg. The rotor stiffness is 49×10^6 N/m and the support stiffness is 10×10^7. Damping is neglected in both the rotor and the support. The six-dof model considered here has been frequently used for studying the lateral vibration of rotors in two perpendicular x and y directions as shown in Fig. 1. The model equations of motion are reported by Abduljabbar et al. [32]. The rotor speed is considered to be 1230 rad/s. The mass, stiffness and damping matrices of this example are shown in Appendix C. In order to satisfy the magnitude condition, we have to isolate the proportional and nonproportional damping parts of the matrix $\mathbf{C}_0$. The proportionality constants are chosen such that $\alpha=0.1$ and $\beta=0.0001$. The solution results are shown in Table 3, where the perturbation approach developed in this paper is still holding a reasonable accuracy in comparison with the exact method.

7. Conclusions

A method is developed to get the general second-order perturbation theory fruitfully applicable to the solution of the eigenvalue problem of nonclassically, viscously damped system. The main contribution here is that the eigensolution of a highly standard eigenvalue problem of single, symmetric positive definite matrix is systematically employed to generate the eigensolution of an asymmetric nonproportionally damped eigenproblem. The later one primarily includes asymmetric damping, stiffness and mass matrices introduced by gyroscopic and circulatory effects. A high compatibility between solutions obtained by the current method and other solutions obtained by exact method is proven.

Appendix A. General perturbation results

Prior to presenting the perturbation results, it is believed that introducing Chung and Lee's [22] matrix formulation and perturbation would be useful to the reader. This is to clarify the difference between the matrix formulation and perturbation used in this paper and that of Ref. [22]. A general viscously damped system was considered in Ref. [22] such that:

$$\mathbf{M}\,\ddot{\mathbf{q}}(t) + \mathbf{C}\,\dot{\mathbf{q}}(t) + \mathbf{K}\,\mathbf{q}(t) = 0 \tag{A.1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n \times n$ real asymmetric matrices. $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, and $\mathbf{q}(t)$ is a real $n \times 1$ vector of generalized coordinates. The matrices $\mathbf{M}$ and $\mathbf{K}$ are assumed positive definite. Note here that the skew symmetric matrices $\mathbf{G}$ and $\mathbf{H}$ are not considered in Chung and Lee's matrix formulation. In Ref. [22], the matrices $\Delta\mathbf{M}$, $\Delta\mathbf{C}$, and $\Delta\mathbf{K}$ are regarded as the mass, damping and stiffness modification matrices such that Eq. (A.1) becomes

$$(\mathbf{M} + \Delta\mathbf{M})\ddot{\mathbf{q}}(t) + (\mathbf{C} + \Delta\mathbf{C})\dot{\mathbf{q}}(t) + (\mathbf{K} + \Delta\mathbf{K})\mathbf{q}(t) = 0 \tag{A.2}$$

When the damping matrix $\mathbf{C}$ is separated into its proportional and nonproportional parts, $\mathbf{C}_p$ and $\mathbf{C}_{np}$, respectively, Eq. (A.2) can be rewritten as follows:

$$(\mathbf{M} + \Delta\mathbf{M})\ddot{\mathbf{q}}(t) + (\mathbf{C}_p + \mathbf{C}_{np} + \Delta\mathbf{C})\dot{\mathbf{q}}(t) + (\mathbf{K} + \Delta\mathbf{K})\mathbf{q}(t) = 0 \tag{A.3}$$

The solution of Eq. (A.3) can be most conveniently solved by transforming the equation to the first-order form, $\mathbf{A}\,\dot{\mathbf{U}} = \mathbf{B}\,\mathbf{U}$, where

$$\mathbf{U} = \{\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{q}^{\mathrm{T}}\} \tag{A.4}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} + \Delta\mathbf{M} & 0 \\ 0 & -\mathbf{K} - \Delta\mathbf{K} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\mathbf{C}_p - \mathbf{C}_{np} - \Delta\mathbf{C} & -\mathbf{K} - \Delta\mathbf{K} \\ -\mathbf{K} - \Delta\mathbf{K} & 0 \end{bmatrix} \tag{A.5}$$

and $\mathbf{A}$ and $\mathbf{B}$ are real symmetric matrices. In the case of weakly nonproportionally damped system being slightly modified, the unperturbed and perturbation matrices are given by

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & -\mathbf{K} \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} -\mathbf{C}_p & -\mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} \tag{A.6}$$

$$\mathbf{A}_1 = \begin{bmatrix} \Delta\mathbf{M} & 0 \\ 0 & -\Delta\mathbf{K} \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} -\mathbf{C}_{np} - \Delta\mathrm{C} & -\Delta\mathbf{K} \\ -\Delta\mathrm{K} & 0 \end{bmatrix} \tag{A.7}$$

Now, it is obvious that there is a great difference between Chung and Lee's matrix formulation of Eqs. (A.4) and (A.5) and the formulation of Eqs (5), (9) and (10) that is considered in this paper. According to Eqs. (A.4) and (A.5), the unperturbed matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ are symmetric and form an eigensystem that is proportionally damped. The matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ are also symmetric with both the nonproportional damping and modification matrices included in them. In our matrix formulation (9) and (10), $\mathbf{A}_0$ and $\mathbf{B}_0$ will be symmetric positive definite and skew symmetric, respectively. While the matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ are skew symmetric and symmetric nonnegative definite, respectively. The reason for suggesting that new matrix formulation of Eqs. (9) and (10) is that the unperturbed and perturbation matrices are either symmetric or skew symmetric. In other words, the skew symmetric matrix $\mathbf{A}_1$ is a perturbation to the symmetric matrix $\mathbf{A}_0$ while the symmetric matrix $\mathbf{B}_1$ is a perturbation to the skew symmetric matrix $\mathbf{B}_0$.

Appendix B. General perturbation results

The perturbation results according to the order of magnitude as derived by Chung and Lee [22] are presented here. These results are similar to those developed by Meirovitch and Ryland [21] except that the second-order perturbation formula is based on the unnormalized eigenvectors instead of the normalized ones:

$$\begin{aligned} \mathbf{O}(0):\ & \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} = 2a_i\delta_{ij} \\ & \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 2a_i\lambda_{0i}\delta_{ij}, \quad i,j = 1,2,\ldots,2n \end{aligned} \tag{B.1}$$

$$\begin{aligned} \mathbf{O}(1):\ & \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{1i} + \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{0i} + \mathbf{v}_{1j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} = 0 \\ & \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{1i} + \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} + \mathbf{v}_{1j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 2a_i\lambda_{1i}\delta_{ij}, \quad i,j = 1,2,\ldots,2n \end{aligned} \tag{B.2}$$

$$\begin{aligned} \mathbf{O}(2):\ & \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{2i} + \mathbf{v}_{oj}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{1i} + \mathbf{v}_{1j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{1i} + \mathbf{v}_{1j}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{0i} + \mathbf{v}_{2j}^{\mathrm{T}}\mathbf{A}_0\mathbf{u}_{0i} = 0 \\ & \mathbf{v}_{0j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{2i} + \mathbf{v}_{oj}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{1i} + \mathbf{v}_{1j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{1i} + \mathbf{v}_{1j}^{\mathrm{T}}\mathbf{B}_1\mathbf{u}_{0i} + \mathbf{v}_{2j}^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_{0i} = 2a_i\lambda_{2i}\delta_{ij} \quad i,j = 1,2,\ldots,2n \end{aligned} \tag{B.3}$$

where $\mathbf{O}(0)$, $\mathbf{O}(1)$ and $\mathbf{O}(2)$ indicate the zero-, first- and second-order perturbation problems, respectively. Note here that Eqs. (B.1) are similar to Eq. (12) of the eigenvalue problem for the unperturbed system. Substituting Eq. (13) into Eq. (B.2), and upon using Eqs. (11) and (B.1), one gets the first-order perturbation solutions:

$$\begin{aligned} & a_j\varepsilon_{ij} = \mathbf{v}_{0j}^{\mathrm{T}}(\mathbf{B}_1 - \lambda_{0i}\mathbf{A}_1)\mathbf{u}_{0i}/2(\lambda_{0i} - \lambda_{0j}) \\ & a_i\gamma_{ij} = \mathbf{v}_{0j}^{\mathrm{T}}(\mathbf{B}_1 - \lambda_{0i}\mathbf{A}_1)\mathbf{u}_{0i}/2(\lambda_{0j} - \lambda_{0i}), \ \ i\neq j, \quad i,j = 1,2,\ldots,2n \end{aligned} \tag{B.4}$$

$$\begin{aligned} & \varepsilon_{ii} = \gamma_{ii} = -\mathbf{v}_{0i}^{\mathrm{T}}\mathbf{A}_1\mathbf{u}_{0i}/4a_i \\ & \lambda_{1i} = \mathbf{v}_{0i}^{\mathrm{T}}(\mathbf{B}_1 - \lambda_{0i}\mathbf{A}_1)\mathbf{u}_{0i}/2a_i, \quad i = j, \quad i = 1,2,\ldots,2n \end{aligned} \tag{B.5}$$

Substituting Eq. (14) into Eq. (B.3), and upon using Eqs. (11), (B.4) and (B.5), one can extract the second-order perturbation solutions. When $i\neq j$,

$$a_j\tilde{\varepsilon}_{ij} = \frac{1}{(\lambda_{0i} - \lambda_{0j})}\left[-a_i\lambda_{1i}\gamma_{ji} - a_j\lambda_{1j}\varepsilon_{ij} + \sum_{k=1}^{2n} a_k\gamma_{jk}\varepsilon_{ik}(\lambda_{0i} - \lambda_{0k})\right] + \sum_{k=1}^{2n}\varepsilon_{ik}\varepsilon_{kj} \tag{B.6}$$

$$a_j\tilde{\varepsilon}_{ij} = \frac{1}{(\lambda_{0j} - \lambda_{0i})}\left[-a_i\lambda_{1i}\gamma_{ji} - a_j\lambda_{1j}\varepsilon_{ij} + \sum_{k=1}^{2n} a_k\gamma_{jk}\varepsilon_{ik}(\lambda_{0j} - \lambda_{0k})\right] + \sum_{k=1}^{2n}\gamma_{jk}\gamma_{ki} \tag{B.7}$$

when $i = j$,

$$\begin{aligned} & \tilde{\varepsilon}_{ii} = \tilde{\gamma}_{ii} = 0.5\sum_{k=1}^{2n}\left[\gamma_{ik}\gamma_{ki} + \varepsilon_{ik}\varepsilon_{ki} + a_k\gamma_{ik}\varepsilon_{ik}/a_i\right], \\ & \lambda_{2i} = \lambda_{1i}(\gamma_{ii} + \varepsilon_{ii}) + \sum_{k=1}^{2n} a_k\gamma_{ik}\varepsilon_{ik}(\lambda_{0i} - \lambda_{0k})/a_i, \quad i = 1,2,\ldots,2n \end{aligned} \tag{B.8}$$

Appendix C. The formulation matrices of Example 3

$$\mathbf{M} = \begin{bmatrix} 153.03 & 0 & 0 & 0 & 0 & 0 \\ 0 & 141.47 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100.80 & 0 & 0 & 0 \\ 0 & 0 & 0 & 153.03 & 0 & 0 \\ 0 & 0 & 0 & 0 & 141.47 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100.80 \end{bmatrix}$$

$$\mathbf{K} = 1.0 \times 10^8 \begin{bmatrix} 0.4900 & -0.4900 & 0 & 0 & 0 & 0 \\ -0.4900 & 1.0842 & -0.5942 & 0 & 0 & 0 \\ 0 & -0.5942 & 0.6942 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4900 & -0.4900 & 0 \\ 0 & 0 & 0 & -0.4900 & 0.8074 & -0.3174 \\ 0 & 0 & 0 & 0 & -0.3174 & 0.4174 \end{bmatrix}$$

$$\mathbf{C} = 1.0 \times 10^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.5244 & -1.5244 & 0 & 0 & 0 \\ 0 & -1.5244 & 1.5244 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8245 & -0.8245 \\ 0 & 0 & 0 & 0 & -0.8246 & 0.8245 \end{bmatrix}$$

$$\mathbf{G} = \mathbf{H} = \underset{6\times 6}{[0]}$$

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